1.1. RING

1.1.1. Introduction

Ring is an algebraic structure. Ring is denoted by R. It follows two operations such as:

1) Addition (+)
2) Multiplication (•)

A ring structure must satisfy certain rules to be a ring. These rules are known as axioms.

Suppose R is a set having two binary operations (+) and (•) then this algebraic structure $<R, +, \cdot>$ is said to form a Ring if:

+ and · is closed on set R, then, $< R, + >$ and $< R, \cdot >$ hold the closure property:

1) $< R, + >$ is an abelian group.
2) $< R, \cdot >$ is a semi group.
3) · is distributed over the binary operation +, i.e.,
   $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ [Left Distribution]
   $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ [Right Distribution]

A ring is defined as a set R. The two binary operations in this set satisfy the following axioms:

1) **Axioms for Addition**
   i) **Closure Law:** For any $a, b \in R$, $(a + b) \in R$.
   ii) **Associative Law:** For any $a, b, c \in R$, $(a + b) + c = a + (b + c)$.
   iii) **Identity Law:** There is an element $0 \in R$ with the property that $(a + 0) = (0 + a) = a \ \forall a \in R$. (The element 0 is called the zero element of R.)
   iv) **Inverse Law:** For any element $a \in R$, there is an element $b \in R$ satisfying $(a + b) = (b + a) = 0$. (We denote this element b by $-a$, and call it the additive inverse or negative of a.)
   v) **Commutative Law:** For any $a, b \in R$, $(a + b) = (b + a)$.
2) **Axioms for Multiplication**
   i) **Closure Law:** For any $a, b \in R$, $ab \in R$.
   ii) **Associative Law:** For any $a, b, c \in R$, $(a.b).c = a.(b.c)$

iii) **Identity Law:** There is an element $1 \in R$ such that $a.1 = 1.a = a \quad \forall a \in R$. (The element 1 is called the identity element of R.)

iv) **Inverse Law:** For any $a \in R$, if $a \neq 0$, then there exists an element $b \in R$ such that $ab = ba = 1$. (We denote this element b by a^{-1}, and call it the multiplicative inverse of a.)

v) **Commutative Law:** For all $a, b \in R$, $(a.b) = (b.a)$.

We can denote the binary operations of the ring by the symbols $*$, $\oplus$, $\odot$, etc in place of '+' and '.'.

Examples

1) The set of natural numbers is not a ring.
2) $(Z, +, \cdot)$ is a commutative ring with unity.
3) $(Q, +, \cdot)$ is a commutative ring with unity.
4) $(R, +, \cdot)$ is a commutative ring with unity.
5) $(C, +, \cdot)$ is a commutative ring with unity.
6) Set R consisting of only 0 elements is a ring under ordinary addition and multiplication.

For example:

The set $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ forms a ring under addition and multiplication modulo 7.

Multiplicative Inverse of an Element

Suppose $(R, +, .)$ is a ring with unit element '1' and a is any element of R.

If $a \cdot b = 1 = b \cdot a$ then this element $b \in R$ is known as multiplicative inverse of a.

a^{-1} is used to denote the multiplicative inverse of an element a.